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THE PERMANENT ROTATIONS OF A BALANCED NON-AUTONOMOUS GYROSTAT†

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The permanent rotations of a gyrostat about its fixed centre of gravity are investigated. It is assumed that the lines of action of the time-dependent gyrostatic momentum vector maintain a constant position in a reference system attached to the carrier body. It is shown that, if the total angular momentum of the gyrostat is non-zero, permanent rotations can only occur about its principal axes of inertia. In that case the gyrostatic momentum vector must be collinear with one of the principal axes of inertia of the gyrostat.

Consider a gyrostat with a fixed centre of gravity, unaffected by any external forces. Assume that the line of action of the time-dependent vector **k**—the angular momentum of the carried bodies of the gyrostat in their motion relative to the carrier body (hereafter: the gyrostatic momentum)—is fixed in a reference system attached to the carrier body. This situation is typical, e.g. when analysing dynamical effects related to the emergency spinning mode of a stationary communications satellite equipped with a three-degree-of-freedom gyroscope with a controlled flywheel (in the spinning mode the gyroscope frames are set against the stops). Thus, $\mathbf{k} = k(t)\mathbf{l}$, where $\mathbf{l}$ is a fixed unit vector in the attached reference system and $k(t)$ is a given function of time, continuous and bounded for all $t \in R$.

The equation of motion of the gyrostat in vector notation in this case has the form

$$\dot{\mathbf{K}} + \boldsymbol{\omega} \times \mathbf{K} = 0 \tag{1}$$

where $\mathbf{K} = \mathbf{I}\boldsymbol{\omega} + \mathbf{k}$ is the total angular momentum of the gyrostat, $\mathbf{I}$ is the tensor of inertia of the gyrostat with respect to its point of attachment, and $\boldsymbol{\omega}$ is the angular velocity vector of the gyrostat. The dot over $\mathbf{K}$ denotes differentiation with respect to time in the attached reference system (relative differentiation).

A first integral of Eq. (1) is

$$\mathbf{K} = \mathbf{I}\boldsymbol{\omega} + \mathbf{k} = \text{const} \tag{2}$$

It is well known that for a body attached at its centre of gravity, the only possible permanent axes of rotation ($\mathbf{k}(t) \equiv 0$) are its principal axes of inertia. For a balanced gyrostat with constant gyrostatic momentum vector ($\mathbf{k}(t) \equiv \text{const}$), the permanent axis may be any generatrix of the cone $(\mathbf{In} \times \mathbf{n}, \mathbf{k}) = 0$, where $\mathbf{n}$ is the unit vector along the permanent axis of rotation in the attached reference system (see, e.g. [1]).

One might expect that in a balanced gyrostat with variable gyrostatic momentum the set of possible permanent rotations would be richer (thus, it being possible to realize an arbitrary vector function $\mathbf{k}(t)$, suitable choice of the latter will enable one to obtain any given permanent rotation of the gyrostat [2]). However, for the case of variable gyrostatic momentum whose line of action is fixed in the attached reference system, if $\mathbf{K} = \mathbf{K}_0 \neq 0$, the following fact may be established.

Theorem. If the total angular momentum is not zero, permanent rotations of a balanced gyrostat, with variable gyrostatic momentum whose line of action is fixed in a reference system attached to the carrier body, are possible only about the gyrostat's principle axes of inertia.

Proof. Express the angular velocity of the gyrostat as $\boldsymbol{\omega} = \omega\mathbf{n}$, where $\mathbf{n}$ is the slope of the permanent axis of rotation.

Since $(\mathbf{In}, \mathbf{n}) > 0$, the vector $\mathbf{In}$ is not perpendicular to the unit vector $\mathbf{n}$ of the axis of rotation, and therefore the plane passing through the vectors $\mathbf{In}$ and $\mathbf{n}$, which are both fixed in the attached reference system, will turn in permanent rotation of the gyrostat about the $\mathbf{n}$ axis, which is not perpendicular to the plane; hence the plane will change position in the inertial reference system. By virtue of the expression

$$\mathbf{K} = \omega \cdot \mathbf{In} + k(t)\mathbf{l} \tag{3}$$

the vector $\mathbf{K} = \mathbf{K}_0$, which is constant in the inertial system, must lie in the plane throughout the rotation of the gyrostat. With the motion of the plane subject to these conditions, this is possible only if the vector $\mathbf{K}$ itself is the axis of rotation, that is, the vectors $\mathbf{K}$ and $\mathbf{n}$ are collinear.

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Thus, for a permanent rotation to be possible we must have $\mathbf{K} = \lambda_1\mathbf{n}$. It then follows from (1) that $\dot{\mathbf{K}} = 0$, that is, in permanent rotation the angular momentum vector of the gyrostat is constant in both inertial and attached reference systems.

Since **K** is constant in the attached reference system, it follows from (3) that if the gyrostatic momentum is variable, permanent rotation can only occur at a variable angular velocity $\omega = \omega(t)$. Since the vector **K** is constant in the attached system, and the directions of the vectors **In** and **n** are also constant in that system, it follows that, if $\omega(t)$ and $k(t)$ are variable, decomposition (3) is only possible if the vectors occurring in the sum on the right coincide in direction, that is, $\mathbf{l} = \lambda_2\mathbf{In}$. Consequently (see (3)), in permanent rotation necessarily $\mathbf{K} = \lambda_3\mathbf{In}$. On the other hand, we have already found that $\mathbf{K} = \lambda_1\mathbf{n}$. Thus, for a permanent rotation to be possible we must have $\mathbf{In} = \lambda\mathbf{n}$ (where λ is a scalar), that is, **n** is an eigenvector of the tensor of inertia **I**.

Corollary. It follows from the proof of the theorem that $\mathbf{l} = \lambda_0\mathbf{n}$. Thus, if the line of action of the gyrostatic momentum vector is fixed in direction, permanent rotation of the gyrostat is only possible if the gyrostatic momentum vector is collinear with one of the gyrostat's principal axes of inertia.

REFERENCES

1. RUMYANTSEV, V. V., The stability of motion of gyrostats. *Prikl. Mat. Mekh.*, 1961, **25**, 1, 9–16.
2. KEIS, I. A., The existence of certain integrals of the equations of motion of a gyrostat attached at a point. *Vestnik Mosk. Gos. Univ., Ser. Mat., Mekh.*, 1963, 6, 55–63.

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